

Acceptability Pricing of Contingent Claims Under Model Ambiguity Using Stochastic Optimization

Martin Glanzer*, Georg Ch. Pflug*

Abstract

Optimal bid and ask prices for contingent claims can be found by mathematical optimization. To do so, a model for the market dynamics is needed. While the traditional replication or superreplication strategies find the optimal prices under the constraint that all risks are shifted to the counterparty, we weaken this assumption here by introducing risk (resp. acceptability) functionals in the stochastic optimization framework. Moreover, we consider the associated ambiguity problem, where we replace the single probability model by a non-parametric set of models. We show that weakening the acceptability constraint leads to a shrinking bid-ask spread while considering model ambiguity makes the bid-ask spread even larger. Some algorithms and numerical examples are presented.

1 Introduction

Initiated by the seminal work of Harrison, Kreps and Pliska [8–10, 17], in the early 1980s a sound theory for the pricing of contingent claims was established based on the no-arbitrage paradigm. While in this early stage the focus was on frictionless and complete market models¹ (such as the celebrated Black-Scholes model), subsequently a plethora of literature emerged, offering generalizations and extensions in several dimensions. In terms of market frictions, for instance, Jouini and Kallal extended the literature regarding transaction costs [12] resp. short-selling constraints [11]. No-arbitrage theory culminated when Delbaen and Schachermayer [4] proved a general version of what is referred to as the *fundamental theorem of asset pricing* (FTAP). In particular, their so-called *superreplication theorem* provides an interval of fair prices for a contingent claim in an incomplete market model². The ask price of the claim is given by the upper bound of this interval, i.e. the supremum of the expected payoff taken over the set of all equivalent martingale measures. It corresponds to shifting any possible risk to the buyer of the claim. Conversely, the lower bound of the interval of fair prices, generated by the infimum of the expected payoff taken over the set of all equivalent martingale measures, gives the bid price of the claim. It amounts to transferring any risk to the writer of the claim. From a practical viewpoint, the

*Department of Statistics and OR (ISOR), University of Vienna, Oskar Morgenstern Platz 1, A-1090 Wien-Vienna Austria

¹Complete market models carry a unique equivalent martingale measure for the discounted asset price process.

²For incomplete market models there exist several (and consequently infinitely many) equivalent martingale measures for the discounted asset price process.

following question arises naturally: Do these price bounds provide meaningful information?

Stimulated by empirical studies and practical observations revealing the fact that market-makers indeed accept controlled risks when setting their spreads, Carr, Geman and Madan [2] introduced a concept to account for bid-ask spreads containing risk. They introduced the broader definition of *acceptable opportunities*, i.e. opportunities that might not be an arbitrage opportunity in the strict sense of its definition, yet any reasonable investor would not doubt its benefits outweighing its costs. Their approach to hedging to acceptability requires the definition of a set of probability measures (*valuation test measures* and *stress test measures*) and a set of non-positive constants (*floors*) associated to the stress test measures. An investment opportunity is then considered acceptable if its expected gain under each valuation measure (reflecting the collective of rational views on the market) is non-negative as well as if under each stress test measure it is greater than or equal to the corresponding floor. Prohibiting acceptable opportunities in this sense yields tighter bid-ask spreads than the classical no-arbitrage price bounds and possibly unique prices in incomplete markets. Driven by a similar motivation, Föllmer and Leukert [6] introduced the technically more involved idea of *quantile hedging*, i.e. maximizing the probability of not falling short given a certain amount of capital. This approach was subsequently generalized to coherent and convex risk measures by Nakano [18, 19] resp. Rudloff [37]. The follow-up paper by Föllmer and Leukert [7] forges a bridge all the way from a perfect superhedge to no hedge by using expectation together with an increasing convex function weighting the possible losses according to the intended strength of the hedge. In contrast, our approach using the Average-Value-at-Risk to control the shortfall risk will allow an agent for a smooth transition between hedging with probability one and hedging w.r.t. expectation by varying the level of the Average-Value-at-Risk between 0 and 1.

Stochastic optimization offers a natural framework to deal with the problems of mathematical finance mentioned above. Application of the fundamental work of Rockafellar [29] and Rockafellar and Wets [30–35] on conjugate duality and stochastic programming has led to a stream of literature which offers several nice properties in comparison with the classical techniques of mathematical finance using sophisticated functional analysis. To review briefly some important contributions in this area, King [15] originally formulated the problem of contingent claim pricing as a stochastic program, already utilizing the flexible structure to investigate extensions such as transaction costs or margin requirements. His investigations reveal that existing liability structures and endowments explain why there is indeed interest in contingent claims from both the buy and the sell side, despite the positive bid-ask spread in incomplete market models. The approach of [15] has been extended, for instance, in King, Koivu and Pennanen [16] to include liquidly traded options for hedging, and in Pennanen and King [22] to the valuation of American type contingent claims. Kallio and Ziemba [14] use the stochastic optimization framework to draw a coherent picture of the effects of market frictions such as transaction costs, short-selling constraints, dividends, or differing interest rates for borrowing and lending, while showing the possibility of rather elementary proofs. Recently, Rognlien Dahl [36] has applied conjugate duality theory to the problem of pricing a contingent claim from the perspective of a writer having only partial information and facing short selling constraints. Summarizing, the striking features of the approach to formulate problems in the

form of convex stochastic programs are as follows: All of the articles mentioned in this paragraph use a discrete time/discrete state space setting; in this setting, pricing measures are attainable. Moreover, whereas the classical functional analytic frameworks lack a certain flexibility, the stochastic programming approach naturally allows for incorporating features and constraints of real-world markets. Most importantly, numerical results can be obtained efficiently by applying the powerful toolkit of available algorithms designed for convex optimization problems.

The well known pricing rule for contingent claims reads (see e.g. King [15] or Pennanen [20, 21]): *The fair ask-price of a contingent claim is the minimal initial capital needed to construct a trading strategy for replicating (or in case this is not feasible) for superreplicating the claim.* This pricing rule is based on the solution of an optimization problem, which is linear (or convex). Its dual formulation reads: *The fair ask-price of a contingent claim equals the maximum of the expected discounted cash-flow out of this contract, where the maximum is taken over all equivalent probability distributions on the state space of the underlying asset price process, which make the discounted underlying asset price process a martingale*³ ('equivalent martingale measures'). There is a similar principle to assess the bid-price of a contingent claim: *The fair bid-price of a contingent claim is the maximal value the holder may borrow at the beginning of the contract such that with the payments out of the contingent claim and an optimal trading strategy she can end up with nonnegative wealth at maturity time.* The dual version of this rule is: *The fair bid-price of a contingent claim equals the minimum of the expected discounted cash-flow out of this contract, where the minimum is taken over all equivalent martingale measures.*

As indicated above, these pricing rules lead in many cases to unrealistically high prices⁴ and in incomplete market models to a positive bid-ask spread, meaning that a trade cannot be explained in such situations. In this paper we extend the classical approach in two ways: In section 2 we replace the almost sure superhedging requirement by the weaker requirement of an acceptable hedge. The acceptability condition is formulated w.r.t. a concrete probability model. This lowers the ask-price and increases the bid-price such that the bid-ask spread may be tightened/closed. In section 3 we weaken the assumption of one single plausible probability model and investigate the effect of model ambiguity on acceptability pricing. Section 4 contains algorithmical aspects and numerical examples.

2 Acceptability pricing

2.1 Acceptability functionals

The notation and definitions introduced in this section are following the book of Pflug and Römisch [27]. A detailed discussion of risk/acceptability func-

³The existence of at least one probability distribution which make the discounted process a martingale is equivalent to the no-arbitrage condition. If there is no such distribution, then both parties, the seller and the buyer of the contract, may make a fortune of arbitrary size by investing in the market and ignoring the contingent claim trade.

⁴For example, the ask price for a plain vanilla call option in exponential Lévy models is given by the spot price of the underlying asset (see [3], Proposition 10.2), which is a trivial upper bound on the call option price.

tionals and their properties can be found therein. For ease of notation, we fix $\mathcal{Y} := L_p(\Omega, \Sigma, \mathbb{P})$, for some $1 \leq p \leq \infty$, for all the definitions below. The corresponding dual space $L_q(\Omega, \Sigma, \mathbb{P})$, such that $\frac{1}{p} + \frac{1}{q} = 1$, is denoted by \mathcal{Z} .

Definition 1. A mapping $\mathcal{A} : \mathcal{Y} \rightarrow \mathbb{R} \cup \pm\infty$ satisfying the three defining properties

(A1) **Translation equivariance:**

$$\mathcal{A}(X + c) = \mathcal{A}(X) + c \quad \forall c \in \mathbb{R},$$

(A2) **Concavity:**

$$\mathcal{A}(\lambda X + (1 - \lambda)Y) \geq \lambda \mathcal{A}(X) + (1 - \lambda)\mathcal{A}(Y) \quad \forall 0 \leq \lambda \leq 1,$$

(A3) **Monotonicity:**

$$X \leq Y \text{ a.s.} \Rightarrow \mathcal{A}(X) \leq \mathcal{A}(Y),$$

for all $X, Y \in \mathcal{Y}$ is called **acceptability functional**.

Definition 2. An acceptability functional \mathcal{A} is called

(i) **positively homogeneous**, if

$$\mathcal{A}(c \cdot Y) = c \cdot \mathcal{A}(Y) \quad \forall c \geq 0.$$

(ii) **proper**, if

$$\mathcal{A}(Y) < +\infty \quad \forall Y \in \mathcal{Y},$$

as well as

$$\{Y \in \mathcal{Y} : \mathcal{A}(Y) > -\infty\} \neq \emptyset.$$

(iii) **upper semicontinuous (u.s.c.)**, if for any $Y \in \mathcal{Y}$ and every sequence $(Y_n)_{n \geq 0}$ such that $Y_n \rightarrow Y$ in L_p , it holds⁵

$$\limsup_n \mathcal{A}(Y_n) \leq \mathcal{A}(Y).$$

The **conjugate** of an acceptability functional is defined as

$$\mathcal{A}^+(Z) := \inf_{Y \in \mathcal{Y}} \{\mathbb{E}[YZ] - \mathcal{A}(Y)\}.$$

It follows directly from the Fenchel-Moreau-Rockafellar Theorem (see Rockafellar [29], Theorem 5), that the biconjugate of a proper u.s.c. acceptability functional equals the functional itself, i.e. any such acceptability functional has a representation of the form

$$\mathcal{A}(Y) := \inf_{Z \in \mathcal{Z}} \{\mathbb{E}[YZ] - \mathcal{A}^+(Z)\}, \quad (1)$$

which is called the **dual representation** of \mathcal{A} . Some useful properties of acceptability functionals and the corresponding conjugate functionals are given by the subsequent proposition (see [27], Theorem 2.31)

⁵For acceptability functionals $\mathcal{A}(Y)$, which depend only on the distribution Y , the upper semicontinuity follows from the concavity, see Jouini, Schachermayer, and Touzi [13].

Proposition 1. *For a proper u.s.c. acceptability functional \mathcal{A} , the following implications hold:*

- 1.) If, for $Z \in \mathcal{Z}$, $\mathcal{A}^+(Z) > -\infty$, then $\mathbb{E}[Z] = 1$.
- 2.) If, for $Z \in \mathcal{Z}$, $\mathbb{P}[Z < 0] > 0$, then $\mathcal{A}^+(Z) = -\infty$.
- 3.) If \mathcal{A} is positively homogeneous, then \mathcal{A}^+ takes only the values 0 or $-\infty$, i.e.

$$\mathcal{A}(Y) = \inf \{ \mathbb{E}[YZ] : Z \in \mathcal{Z}_{\mathcal{A}} \},$$

where $\mathcal{Z}_{\mathcal{A}} = \{Z \in \mathcal{Z} : \mathcal{A}^+(Z) > -\infty\}$ is the supergradient set of \mathcal{A} .

Corollary 1. *A proper u.s.c. acceptability functional \mathcal{A} has the dual representation*

$$\mathcal{A}(Y) = \inf \{ \mathbb{E}[YZ] - \mathcal{A}^+(Z) : Z \in \mathcal{Z}, Z \geq 0 \text{ a.s.}, \mathbb{E}[Z] = 1 \}.$$

There is a (partial) ordering on the collection of acceptability functionals and a minimal as well as a maximal element exist. The ordering can be characterized in terms of the dual sets:

$$\mathcal{A}^{(1)} \leq \mathcal{A}^{(2)} : \Longleftrightarrow \mathcal{Z}_{\mathcal{A}^{(2)}} \subseteq \mathcal{Z}_{\mathcal{A}^{(1)}}.$$

The extreme cases are represented by the essential infimum ($\mathcal{A} = \text{essinf} \Leftrightarrow \mathcal{Z}_{\mathcal{A}} = \{Z : Z \geq 0 \text{ a.s.}, \mathbb{E}[Z] = 1\}$) and the expected value ($\mathcal{A} = \mathbb{E} \Leftrightarrow \mathcal{Z}_{\mathcal{A}} = \{1\}$). In fact, the Average-Value-at-Risk is a representative of the collection of acceptability functionals that describes the whole spectrum. The Average-Value-at-Risk (also referred to as Expected Shortfall, Conditional-Value-at-Risk, or Tail-Value-at-Risk) at some level $0 < \alpha \leq 1$ is defined as

$$\text{AV@R}_{\alpha}(Y) := \frac{1}{\alpha} \int_0^{\alpha} G_Y^{-1}(u) du,$$

where G_Y^{-1} denotes the inverse distribution function of some random variable $Y \in \mathcal{Y}$. It holds that $\text{AV@R}_1(Y) = \mathbb{E}[Y]$ and $\text{AV@R}_{\alpha}(Y) \rightarrow \text{essinf}(Y)$ for $\alpha \downarrow 0$. Moreover, the Average-Value-at-Risk is monotonically increasing in α on $(0, 1]$. It is a concave minorant of the Value-at-Risk, which makes it a tractable yet well-interpretable alternative to the popular but nonconvex Value-at-Risk. Therefore, all the numerical studies in section 4 will be based on the AV@R.

2.2 Replication to acceptability

Let us now introduce the notion of acceptability in the pricing procedure for contingent claims.

In the mathematical finance literature, a market model is given by some filtered probability space, i.e. a scenario space Ω , a filtration \mathcal{F} , and a probability distribution \mathbb{P} . On $(\Omega, \mathcal{F}, \mathbb{P})$ there lives a $(d+1)$ -dimensional \mathcal{F} -adapted price process $S_t = (S_t^{(0)}, S_t^{(1)}, \dots, S_t^{(d)})$ representing liquidly traded basic assets. One asset, denoted by $S_t^{(0)}$, is strictly positive (a.s.) with $S_0^{(0)} = 1$, and serves as numéraire for discounting. Often, a riskless bond is chosen as a numéraire. We work in a discrete-time market model, i.e. we assume finitely many time

points $t = 0, \dots, T$ where trading activity may occur. Let the filtration \mathcal{F} be the σ -algebra generated by S . A contingent claim C consists of an \mathcal{F} -adapted series of cash flows C_1, \dots, C_T , i.e. the payoff C_t is contingent on the respective state of the market at time $1 \leq t \leq T$. A trading strategy is an \mathcal{F} -adapted $(d+1)$ -dimensional real valued process.

Definition 3. Let $(\Omega, \mathcal{F}, \mathbb{P}, S)$ some discrete-time market model. For each trading time $t = 1, \dots, T$ fix a proper u.s.c. acceptability functional \mathcal{A}_t . Then, for a contingent claim C with Lipschitz payoffs $C_1 := C_1(S_1), \dots, C_T := C_T(S_T)$,

(i) the **acceptable ask-price** is defined as

$$\begin{aligned} \pi_a(\mathcal{A}_1, \dots, \mathcal{A}_T) &:= \min_{x, w} w \\ \text{s.t. } &x_0^\top S_0 \leq w \\ &\mathcal{A}_t(x_{t-1}^\top S_t - x_t^\top S_t - C_t) \geq 0 \quad \forall t = 1, \dots, T-1; \\ &\mathcal{A}_T(x_{T-1}^\top S_T - C_T) \geq 0, \end{aligned} \quad (2)$$

(ii) the **acceptable bid-price** is defined as

$$\begin{aligned} \pi_b(\mathcal{A}_1, \dots, \mathcal{A}_T) &:= \max_{x, w} w \\ \text{s.t. } &x_0^\top S_0 \geq w \\ &\mathcal{A}_t(x_t^\top S_t - x_{t-1}^\top S_t + C_t) \geq 0 \quad \forall t = 1, \dots, T-1; \\ &\mathcal{A}_T(-x_{T-1}^\top S_T + C_T) \geq 0, \end{aligned} \quad (3)$$

where the optimization runs over $w \in \mathbb{R}$ and trading strategies x for the liquidly traded assets.

In order to exclude pathological cases (in particular, (a.s.) linearly dependent basic assets shall be prevented), we introduce an additional (and rather natural) condition on the market model. Define first

$$Y_t^x := \begin{cases} x_{t-1}^\top S_t - x_t^\top S_t - C_t & t = 1, \dots, T-1 \\ x_{t-1}^\top S_t - C_t & t = T. \end{cases}$$

Condition C. For $t = 1, \dots, T$, all level sets $L_c^t = \{x : \mathcal{A}_t(Y_t^x) \geq c\}$ of the mapping $x_t \mapsto \mathcal{A}_t(Y_t^x)$ are compact (or empty).

In the sequel, we always assume that Condition C is respected by the price processes of the basic assets.

Proposition 2. Let \mathcal{A}_t be a positively homogeneous proper u.s.c. acceptability functional with supergradient set $\mathcal{Z}_{\mathcal{A}_t}$, for all $t = 1, \dots, T$. Then, problem (2) is dual (in the strong sense) to problem (4), i.e. it holds

$$\begin{aligned} \pi_a(\mathcal{A}_1, \dots, \mathcal{A}_T) &= \sup_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}} \left[\sum_{t=1}^T \tilde{C}_t \right] \\ \text{s.t. } &\mathbb{E}^{\mathbb{Q}}[\tilde{S}_{t+1} | \mathcal{F}_t] = \tilde{S}_t \quad \forall t = 0, \dots, T-1; \\ &\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} \in \mathcal{Z}_{\mathcal{A}_t} \quad \forall t = 1, \dots, T. \end{aligned} \quad (4)$$

Before we move on with the proof of Proposition 2, we shall state two Lemmas.

Lemma 1 (Bayes' formula for conditional expectation). *Let $(\Omega, \Sigma, \mathbb{P}, \mathcal{F})$ be a filtered probability space. Let \mathbb{Q} a probability measure such that $\mathbb{Q} \sim \mathbb{P}$, and let $X \in L^1(\mathbb{Q})$. Then*

$$\mathbb{E}^{\mathbb{Q}}[X|\mathcal{F}] = \frac{\mathbb{E}^{\mathbb{P}}[X \frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}]}{\mathbb{E}^{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}]} \quad \text{a.s.}$$

Proof. Define $Z := \frac{d\mathbb{Q}}{d\mathbb{P}}$, which exists and is a \mathbb{P} -integrable nonnegative random variable according to the Radon-Nikodým theorem. Moreover, define $Y := \frac{\mathbb{E}^{\mathbb{P}}[XZ|\mathcal{F}]}{\mathbb{E}^{\mathbb{P}}[Z|\mathcal{F}]}$. Then, for $F \in \mathcal{F}$, we obtain

$$\begin{aligned} \int_F Y d\mathbb{Q} &= \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_F Y] = \mathbb{E}^{\mathbb{P}}[\mathbf{1}_F Y Z] \\ &= \mathbb{E}^{\mathbb{P}}[\mathbf{1}_F Y \mathbb{E}^{\mathbb{P}}[Z|\mathcal{F}]] = \mathbb{E}^{\mathbb{P}}[\mathbf{1}_F \mathbb{E}^{\mathbb{P}}[XZ|\mathcal{F}]] \\ &= \mathbb{E}^{\mathbb{P}}[\mathbf{1}_F X Z] = \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_F X] \\ &= \int_F X d\mathbb{Q} \end{aligned}$$

Thus, by the definition of conditional expectation, $Y = \mathbb{E}[X|\mathcal{F}]$ holds a.s. \square

Now let us consider a stochastic optimization problem of the form

$$v^* = \min\{c(x) : \mathcal{A}(Y_x) \geq 0\},$$

where x is a vector in \mathbb{R}^N , $x \mapsto c(x)$ is a continuous function $\mathbb{R}^N \rightarrow \mathbb{R}$, and $x \mapsto Y_x$ is a continuous function $\mathbb{R}^N \rightarrow L_p(\mathbb{R}^m, P)$. The functional \mathcal{A} is assumed to have a representation of the form

$$\mathcal{A}(Y) = \inf\{\mathbb{E}(Y \cdot Z) : Z \in \mathcal{Z}\},$$

where \mathcal{Z} is a closed convex set in $L_q(\mathbb{R}^m, P)$, for $\frac{1}{p} + \frac{1}{q} = 1$. We assume that $v^* > -\infty$. Since L_q is separable, there is a sequence (Z_i) such that $\{Z_1, Z_2, \dots\}$ is dense in \mathcal{Z} . Let $\mathcal{Z}_n = \text{conv}\{Z_1, \dots, Z_n\}$ and let

$$\mathcal{A}_n(Y) = \min\{\mathbb{E}(Y \cdot Z) : Z \in \mathcal{Z}_n\}.$$

Here is our assertion.

Lemma 2. *Let $v_n := \min\{c(x) : \mathcal{A}_n(Y_x) \geq 0\}$. Then $v_n \uparrow v^*$.*

Proof. Since $\mathcal{A}_n(Y) \geq \mathcal{A}(Y)$, we have that $D_n = \{x : \mathcal{A}_n(Y_x) \geq 0\}$ is a monotonically decreasing sequence of sets. Because of Condition C, D_1 is compact. Obviously, v_n is a monotonically increasing sequence. We have to show that $\lim_n v_n$ cannot be smaller than v^* . Let $x_n^* \in \text{argmin}\{c(x) : \mathcal{A}_n(Y_x) \geq 0\}$. Let x^* be an accumulation point of (x_n^*) , i.e. $x_{n_i}^* \rightarrow x^*$. We show that $x^* \in D = \{x : \mathcal{A}(Y_x) \geq 0\}$. Suppose the contrary. Then $\mathcal{A}(Y_{x^*}) < 0$, i.e. there is a $Z^* \in \mathcal{Z}$ such that $\mathbb{E}(Y_{x^*} \cdot Z^*) < 0$. However, there is a sequence $(Z_{n_i}') \in \mathcal{Z}_{n_i}'$ such that $Z_{n_i}' \in \mathcal{Z}_{n_i}'$ and $Z_{n_i}' \rightarrow Z^*$. Since $\mathbb{E}(Y_{x_n^*} \cdot Z_{n_i}') \geq 0$ for all n_i' , this contradiction implies that $\mathcal{A}(Y_{x^*}) \geq 0$. Obviously (v_n) is a monotonically increasing

sequence with $v_n \leq v^*$. On the other hand, $v^* \leq c(x^*) = \lim_i c(x_{n_i}) = \lim_i v_{n_i}$. Therefore, $\limsup v_n = v^*$ and by monotonicity,

$$\lim_n v_n = v^*.$$

□

Proof(Proposition 2). By assumption, each acceptability functional \mathcal{A}_t is proper and u.s.c. Thus, by Corollary 1, a representation of the form

$$\mathcal{A}_t(Y) = \inf \{ \mathbb{E}[YZ] - \mathcal{A}_t^+(Z) : Z \in \mathcal{Z}_t, Z \geq 0 \text{ a.s.}, \mathbb{E}[Z] = 1 \}, \quad (5)$$

exists. Moreover, \mathcal{A}_t is assumed to be positively homogeneous. By Proposition 1, there exists a supergradient set $\mathcal{Z}_{\mathcal{A}_t}$ such that

$$\mathcal{A}_t(Y) = \inf \{ \mathbb{E}[YZ] : Z \in \mathcal{Z}_{\mathcal{A}_t} \}.$$

By Lemma 2, the optimal value of (2) can be approximated by the solution of a similar program, but $\mathcal{A}_t(x_{t-1}^\top S_t - x_t^\top S_t - C_t)$ replaced by a finitely generated functional

$$\inf \{ \mathbb{E}[(x_{t-1}^\top S_t - x_t^\top S_t - C_t) \cdot Z : Z \in \mathcal{Z}_{\mathcal{A}_t}^{n_t}] \},$$

where

$$\mathcal{Z}_{\mathcal{A}_t}^{n_t} = \text{conv} \{ Z_t^i : i = 1, \dots, n_t \}.$$

More precisely, problem (2) is approximated by

$$\begin{aligned} \pi_a^n &:= \min_{x, w} w \\ \text{s.t. } & x_0^\top S_0 \leq w \\ & \mathbb{E}[(x_{t-1}^\top S_t - x_t^\top S_t - C_t) \cdot Z_t^i] \geq 0 \quad \forall i = 1, \dots, n_t; \forall t = 1, \dots, T-1; \\ & \mathbb{E}[(x_{T-1}^\top S_T - C_T) \cdot Z_T^i] \geq 0 \quad \forall i = 1, \dots, n_T. \end{aligned}$$

Considering the fact that this is a convex optimization problem, there is no duality gap and the Lagrange dual problem can be formulated as

$$\begin{aligned} & \sup_{\lambda_0 \geq 0} \inf_{x, w} w + \lambda_0(x_0^\top S_0 - w) \\ & \quad - \sum_{t=1}^{T-1} \sum_{i=1}^{n_t} \lambda_t^i (\mathbb{E}[Z_t^i \cdot (x_{t-1}^\top S_t - x_t^\top S_t - C_t)]) \\ & \quad - \sum_{i=1}^{n_T} \lambda_T^i (\mathbb{E}[Z_T^i \cdot (x_{T-1}^\top S_T - C_T)]). \end{aligned}$$

Some rearrangements yield the representation

$$\begin{aligned} & \sup_{\lambda_0 \geq 0} \inf_{x, w} w(1 - \lambda_0) + x_0^\top \left(\lambda_0 S_0 - \mathbb{E} \left[S_1 \left(\sum_{i=1}^{n_1} \lambda_1^i Z_1^i \right) \right] \right) \\ & \quad + \sum_{t=1}^{T-1} \mathbb{E} \left[x_t^\top \left(S_t \left(\sum_{i=1}^{n_t} \lambda_t^i Z_t^i \right) - \mathbb{E} \left[S_{t+1} \left(\sum_{i=1}^{n_{t+1}} \lambda_{t+1}^i Z_{t+1}^i \right) \middle| \mathcal{F}_t \right] \right) \right] \\ & \quad + \sum_{t=1}^T \mathbb{E} \left[C_t \left(\sum_{i=1}^{n_t} \lambda_t^i Z_t^i \right) \right]. \end{aligned} \quad (6)$$

Setting

$$W_t := \sum_{i=1}^{n_t} \lambda_t^i Z_t^i,$$

and carrying out explicitly the inner optimization in w and x_t gives the conditions

$$\lambda_0 = 1, \tag{7}$$

and

$$S_t W_t = \mathbb{E}[S_{t+1} W_{t+1} | \mathcal{F}_t], \tag{8}$$

for all $t = 0, \dots, T-1$, respectively. Condition (8) follows from the fact that

$$\inf_{x_t} \mathbb{E} \left[x_t^\top (S_t W_t - \mathbb{E}[S_{t+1} W_{t+1} | \mathcal{F}_t]) \right] = \begin{cases} 0 & \text{if } (S_t W_t - \mathbb{E}[S_{t+1} W_{t+1} | \mathcal{F}_t]) = 0 \text{ a.s.} \\ -\infty & \text{else.} \end{cases}$$

For the sake of boundedness of the problem, the first case remains as a necessary condition. The unconstrained maximin problem (6) can then be written in the form of a constrained maximization problem:

$$\begin{aligned} \pi_a^n = \sup_{W_t} \mathbb{E} \left[\sum_{t=1}^T C_t W_t \right] \\ \text{s.t. } S_t W_t = \mathbb{E}[S_{t+1} W_{t+1} | \mathcal{F}_t] \quad \forall t = 0, \dots, T \\ W_t \in \text{span}\{Z_t^i\} \quad \forall t = 1, \dots, T. \end{aligned} \tag{9}$$

Recall that S_t is a $(d+1)$ -dimensional asset price process, where the first component $S_t^{(0)}$ represents the numéraire. Define $\tilde{S}_t := S_t / S_t^{(0)}$ and $\gamma_t := W_t S_t^{(0)}$. Using this notation, the constraints in (9) can equivalently be written as

$$\mathbb{E}[\gamma_{t+1} \tilde{S}_{t+1} | \mathcal{F}_t] = \gamma_t \tilde{S}_t, \tag{10}$$

or, in words, that the process $(\gamma_t \tilde{S}_t)$ is a $(d+1)$ -dimensional martingale. In particular, this means that

$$\mathbb{E}[\gamma_{t+1} | \mathcal{F}_t] = \mathbb{E}[\gamma_{t+1} \tilde{S}_{t+1}^{(0)} | \mathcal{F}_t] = \gamma_t \tilde{S}_t^{(0)} = \gamma_t, \tag{11}$$

i.e. the processes (γ_t) itself is required to be a martingale.

Now, let us introduce a new probability measure \mathbb{Q}_γ given by the Radon-Nikodým derivative $\frac{d\mathbb{Q}_\gamma}{d\mathbb{P}} = \gamma_T$. The measure \mathbb{Q}_γ is indeed a probability measure, since

$$\mathbb{Q}_\gamma(\Omega) = \int_{\Omega} \gamma_T d\mathbb{P} = \mathbb{E}^\mathbb{P}[\gamma_T] = \gamma_0 = \lambda_0 S_0^{(0)} = 1.$$

Let us now formulate the martingale condition in terms of \mathbb{Q}_γ . Using Lemma

1, we obtain

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_\gamma} [\tilde{S}_{t+1} | \mathcal{F}_t] &= \frac{\mathbb{E}^\mathbb{P} [\tilde{S}_{t+1} \gamma_T | \mathcal{F}_t]}{\mathbb{E}^\mathbb{P} [\gamma_T | \mathcal{F}_t]} \\
&= \frac{\mathbb{E}^\mathbb{P} [\mathbb{E}^\mathbb{P} [\tilde{S}_{t+1} \gamma_T | \mathcal{F}_{t+1}] | \mathcal{F}_t]}{\gamma_t} \\
&= \frac{\mathbb{E}^\mathbb{P} [\tilde{S}_{t+1} \gamma_{t+1} | \mathcal{F}_t]}{\gamma_t} \\
&= \tilde{S}_t.
\end{aligned}$$

Hence, we see that the martingale condition on $(S_t W_t)$ in terms of \mathbb{P} is equivalent to the condition that (\tilde{S}_t) is a martingale w.r.t. \mathbb{Q}_γ and $\frac{d\mathbb{Q}_\gamma}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \gamma_t$.

Moreover, rewriting the objective function in (9) in terms of \mathbb{Q}_γ gives

$$\begin{aligned}
\mathbb{E}^\mathbb{P} \left[\sum_{t=1}^T C_t W_t \right] &= \mathbb{E}^{\mathbb{Q}_\gamma} \left[\sum_{t=1}^T C_t W_t \frac{1}{\gamma_T} \right] \\
&= \mathbb{E}^{\mathbb{Q}_\gamma} \left[\sum_{t=1}^T \frac{C_t}{S_t^{(0)}} \gamma_t \mathbb{E}^{\mathbb{Q}_\gamma} \left[\frac{1}{\gamma_T} | \mathcal{F}_t \right] \right],
\end{aligned}$$

where, by Lemma 1, the inner expectation is given by

$$\mathbb{E}^{\mathbb{Q}_\gamma} \left[\frac{1}{\gamma_T} | \mathcal{F}_t \right] = \frac{\mathbb{E}^\mathbb{P} \left[\frac{1}{\gamma_T} \gamma_T | \mathcal{F}_t \right]}{\mathbb{E}^\mathbb{P} [\gamma_T | \mathcal{F}_t]} = \frac{1}{\gamma_t},$$

and hence

$$\mathbb{E}^\mathbb{P} \left[\sum_{t=1}^T C_t W_t \right] = \mathbb{E}^{\mathbb{Q}_\gamma} \left[\sum_{t=1}^T \tilde{C}_t \right],$$

where \tilde{C}_t denotes the payment C_t w.r.t. the numéraire $S_t^{(0)}$. Combining the above results, problem (9) can be rewritten in terms of $\mathbb{Q} := \mathbb{Q}_\gamma$ as

$$\begin{aligned}
\pi_a^n &= \sup_{\mathbb{Q}} \mathbb{E}^\mathbb{Q} \left[\sum_{t=1}^T \tilde{C}_t \right] \\
\text{s.t. } &\mathbb{E}^\mathbb{Q} [\tilde{S}_{t+1} | \mathcal{F}_t] = \tilde{S}_t, \quad \forall t = 0, \dots, T-1 \\
&\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \gamma_t \quad \forall t = 1, \dots, T.
\end{aligned} \tag{12}$$

Using

$$1 = \mathbb{E}^\mathbb{P} [\gamma_t] = \mathbb{E}^\mathbb{P} \left[S_t^{(0)} \sum_{i=1}^{n_t} \lambda_t^i Z_t^i \right] = S_t^{(0)} \sum_{i=1}^{n_t} \lambda_t^i,$$

the condition on the measure change can be reformulated as

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = \sum_{i=1}^{n_t} \lambda_t^i Z_t^i \quad \text{subject to} \quad \sum_{i=1}^{n_t} \lambda_t^i = 1,$$

or

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_t} \in \text{conv}\{Z_t^i\} = \mathcal{Z}_{\mathcal{A}_t}^{n_t}.$$

It is left to show that there is no duality gap in the limit $n \rightarrow \infty$. Assume that the dual problem

$$\begin{aligned} & \sup_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}} \left[\sum_{t=1}^T \tilde{C}_t \right] \\ & \text{s.t. } \mathbb{E}^{\mathbb{Q}} \left[\tilde{S}_{t+1} \middle| \mathcal{F}_t \right] = \tilde{S}_t, \quad \forall t = 0, \dots, T-1 \\ & \left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_t} \in \mathcal{Z}_{\mathcal{A}_t} \quad \forall t = 1, \dots, T. \end{aligned}$$

has an optimal value $\pi'_a \neq \pi_a$. For any feasible solution \mathbb{Q} of the dual problem, it holds

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}} \left[\sum_{t=1}^T \tilde{C}_t \right] &= \mathbb{E}^{\mathbb{P}} \left[\sum_{t=1}^T \tilde{C}_t Z_t \right] \\ &\leq \mathbb{E}^{\mathbb{P}} \left[\sum_{t=1}^{T-1} (x_{t-1}^\top \tilde{S}_t - x_t^\top \tilde{S}_t) \cdot Z_t + x_{T-1}^\top \tilde{S}_T Z_T \right] \end{aligned}$$

by the primal constraints. Elaborating this relation, we obtain

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}} \left[\sum_{t=1}^T \tilde{C}_t \right] &\leq \mathbb{E}^{\mathbb{P}} \left[x_0^\top \tilde{S}_1 Z_1 \right] + \sum_{t=1}^{T-1} \mathbb{E}^{\mathbb{P}} \left[x_t^\top (\tilde{S}_{t+1} Z_{t+1} - \tilde{S}_t Z_t) \right] \\ &= x_0^\top \tilde{S}_0 + \sum_{t=1}^{T-1} \mathbb{E}^{\mathbb{P}} \left[\mathbb{E}^{\mathbb{P}} \left[x_t^\top (\tilde{S}_{t+1} Z_{t+1} - \tilde{S}_t Z_t) \middle| \mathcal{F}_t \right] \right] \\ &= x_0^\top \tilde{S}_0 + \sum_{t=1}^{T-1} \mathbb{E}^{\mathbb{P}} \left[x_t^\top (\tilde{S}_t Z_t - \tilde{S}_t Z_t) \right] \\ &= x_0^\top \tilde{S}_0 \\ &\leq w. \end{aligned}$$

Thus, the optimal primal solution π_a is also greater than or equal to the optimal dual solution π'_a . Now assume $\pi'_a < \pi_a$. Then, since $\pi_a^n \uparrow \pi_a$ by Lemma 2, there must exist some n such that $\pi_a^n > \pi'_a$. Moreover, there exists some \mathbb{Q}^n , which is dual feasible and such that $\mathbb{E}^{\mathbb{Q}^n} \left[\sum_{t=1}^T \tilde{C}_t \right] = \pi_a^n$. This is a contradiction to π'_a being the limit of the monotonically increasing sequence of optimal values of the approximate dual problems of the form (12). Hence, $\pi'_a = \pi_a$, i.e. it is shown that there is no duality gap in the limit. \square

Proposition 3. *Let \mathcal{A}_t be a positively homogeneous proper u.s.c. acceptability functional with supergradient set $\mathcal{Z}_{\mathcal{A}_t}$, for all $t = 1, \dots, T$. Then, problem (3) is dual (in the strong sense) to problem (13), i.e. it holds*

$$\begin{aligned} \pi_b(\mathcal{A}_1, \dots, \mathcal{A}_T) &= \inf_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}} \left[\sum_{t=1}^T \tilde{C}_t \right] \\ \text{s.t. } \mathbb{E}^{\mathbb{Q}}[\tilde{S}_{t+1}|\mathcal{F}_t] &= \tilde{S}_t \quad \forall t = 0, \dots, T-1; \\ \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} &\in \mathcal{Z}_{\mathcal{A}_t} \quad \forall t = 1, \dots, T. \end{aligned} \quad (13)$$

Proof. Analogous to the proof of Proposition 2. □

Remark 1. If $C_t \equiv 0$ for some time $1 \leq t \leq T$, a reasonable formulation of the problem will require the hedging strategy to be self-financing in this point, which corresponds to the constraint $x_{t-1}^\top S_t - x_t^\top S_t = 0$ \mathbb{P} -a.s. In particular, for the important special case of European type claims, i.e. claims where there is a payoff only at the time of maturity T , this means that the acceptability super- and subhedging problems read

$$\begin{aligned} \min_{x, w} \quad & w \\ \text{s.t. } \quad & x_0^\top S_0 \leq w \\ & x_{t-1}^\top S_t - x_t^\top S_t \geq 0 \quad \text{a.s. } \forall t = 1, \dots, T-1; \\ & \mathcal{A}_T(x_{T-1}^\top S_T - C_T) \geq 0 \end{aligned}$$

and

$$\begin{aligned} \max_{x, w} \quad & w \\ \text{s.t. } \quad & x_0^\top S_0 \geq w \\ & x_{t-1}^\top S_t - x_t^\top S_t \leq 0 \quad \text{a.s. } \forall t = 1, \dots, T-1; \\ & \mathcal{A}_T(C_T - x_{T-1}^\top S_T) \geq 0, \end{aligned}$$

respectively. This is a special case of the primal problem formulations (2) and (3), where $\mathcal{A}_t = \text{essinf}$ for all $t = 1, \dots, T-1$. The subsequent corollary contains the dual problems.

Corollary 2. *Consider some contingent claim of the European type with payoff C_T at maturity. Assume that buyers and sellers of claims accept a hedging shortfall which is expressed by the acceptability functional \mathcal{A} having a representation of the form $\mathcal{A}(Y) := \inf \{ \mathbb{E}[YZ] : Z \in \mathcal{Z}_{\mathcal{A}} \}$. Define*

$$\mathcal{Q}^{\mathcal{A}} := \left\{ \mathbb{Q} : \mathbb{Q} \sim \mathbb{P}; \mathbb{E}^{\mathbb{Q}}[\tilde{S}_{t+1}|\mathcal{F}_t] = \tilde{S}_t \quad \forall t = 0, \dots, T-1; \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_T} \in \mathcal{Z}_{\mathcal{A}} \right\}. \quad (14)$$

Then, the spread made up by the acceptable bid and ask price is given by

$$\left[\pi_b(\text{essinf}, \dots, \text{essinf}, \mathcal{A}_T) = \inf_{\mathbb{Q} \in \mathcal{Q}^{\mathcal{A}}} \mathbb{E}^{\mathbb{Q}}[\tilde{C}_T], \pi_a(\text{essinf}, \dots, \text{essinf}, \mathcal{A}_T) = \sup_{\mathbb{Q} \in \mathcal{Q}^{\mathcal{A}}} \mathbb{E}^{\mathbb{Q}}[\tilde{C}_T] \right].$$

Remark 2. Corollary 2 corresponds to a generalization of the classic result with pointwise hedging. The corresponding dual set \mathcal{Z} to the acceptability functional $\mathcal{A}(Y) = \text{essinf}(Y)$ is given by the set of all nonnegative random variables. Thus, by choosing $\mathcal{A}(Y) = \text{essinf}(Y)$, there is no (additional) constraint on the measure change $\frac{d\mathbb{Q}}{d\mathbb{P}}$ in (14) and the classic result can be recovered.

Remark 3. The weakest choice for an acceptability functional, i.e. the contrary extreme to $\mathcal{A}(Y) = \text{essinf}(Y)$, would be $\mathcal{A}(Y) = \mathbb{E}[Y]$. Then, $\mathcal{Z} = \{Z : Z = 1 \text{ } \mathbb{P}\text{-a.s.}\}$, which corresponds to the condition $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_T} = 1$ in (14), i.e. $\mathbb{Q}|_{\mathcal{F}_T} = \mathbb{P}|_{\mathcal{F}_T}$ a.s. Consequently, (in case \mathbb{Q} is feasible) the acceptable bid and ask price coincide and the resulting (unique) price of the (European type) contract is given by $\mathbb{E}^{\mathbb{P}}[\tilde{C}_T]$. However, typically both parties (i.e. buyer and seller) will consider a contract on the basis of this price as too risky to agree on it.

3 Acceptability pricing under ambiguity

In section 2.2 we defined the bid/ask price of some contingent claim as the maximal/minimal amount of capital needed in order to sub/superreplicate its payoff(s) w.r.t. to an acceptability measure. However, the result computed with this approach heavily depends on the particular choice of the probability model. Therefore, in this section, the strong model dependency shall be weakened. More specifically, acceptability bid and ask prices shall be robust w.r.t. all models contained in a certain ambiguity set, i.e. a set of plausible models. Our particular interest is in parameter-free ambiguity sets around some (estimated) baseline model.

Definition 4. Define proper u.s.c. acceptability functionals \mathcal{A}_t , $t = 1, \dots, T$, and an ambiguity set \mathcal{P}_ϵ of probability models. Then, for a contingent claim C with Lipschitz payoffs $C_1 := C_1(S_1), \dots, C_T := C_T(S_T)$,

(i) the **robust acceptable ask-price** is defined as

$$\begin{aligned} \pi_a^{\mathcal{P}_\epsilon}(\mathcal{A}_1, \dots, \mathcal{A}_T) &:= \min_{x, w} w \\ \text{s.t. } &x_0^\top S_0 \leq w \\ &\mathcal{A}_t^\mathbb{P}(x_{t-1}^\top S_t - x_t^\top S_t - C_t) \geq 0 \quad \forall \mathbb{P} \in \mathcal{P}_\epsilon; \forall t = 1, \dots, T-1 \\ &\mathcal{A}_T^\mathbb{P}(x_{T-1}^\top S_T - C_T) \geq 0 \quad \forall \mathbb{P} \in \mathcal{P}_\epsilon, \end{aligned} \tag{15}$$

(ii) the **robust acceptable bid-price** is defined as

$$\begin{aligned} \pi_b^{\mathcal{P}_\epsilon}(\mathcal{A}_1, \dots, \mathcal{A}_T) &:= \max_{x, w} w \\ \text{s.t. } &x_0^\top S_0 \geq w \\ &\mathcal{A}_t^\mathbb{P}(x_t^\top S_t - x_{t-1}^\top S_t + C_t) \geq 0 \quad \forall \mathbb{P} \in \mathcal{P}_\epsilon; \forall t = 1, \dots, T-1 \\ &\mathcal{A}_T^\mathbb{P}(-x_{T-1}^\top S_T + C_T) \geq 0 \quad \forall \mathbb{P} \in \mathcal{P}_\epsilon, \end{aligned} \tag{16}$$

where the optimization runs over $w \in \mathbb{R}$ and trading strategies x for the liquidly traded assets. For any $C_t \equiv 0$, $1 \leq t \leq T$, we set $\mathcal{A}_t := \text{essinf}$ in (15) and (16).

Proposition 4. Consider a finite ambiguity set of models $\mathcal{P}_\epsilon = \{\mathbb{P}_1, \dots, \mathbb{P}_n\}$. Then, problem (15) is dual (in the strong sense) to problem (17), i.e. it holds

$$\begin{aligned} \pi_a^{\mathcal{P}}(\mathcal{A}_1, \dots, \mathcal{A}_T) &= \sup_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}} \left[\sum_{t=1}^T \tilde{C}_t \right] \\ \text{s.t. } \mathbb{E}^{\mathbb{Q}} \left[\tilde{S}_{t+1} \middle| \mathcal{F}_t \right] &= \tilde{S}_t \quad \forall t = 0, \dots, T-1 \\ \frac{d\mathbb{Q}}{d\hat{\mathbb{P}}} \bigg|_{\mathcal{F}_t} &\in \text{conv} \left\{ Z_t^{i,j} f_t^j \right\} \quad \forall t = 1, \dots, T, \end{aligned} \quad (17)$$

where $\hat{\mathbb{P}}$ is a universal model such that all $\mathbb{P}^0, j = 1, \dots, n$, are absolutely continuous with density $\frac{d\mathbb{P}^0}{d\hat{\mathbb{P}}} = f_j$, and $Z_t^{i,j}, i \in I_t^j$, form the dual set associated with $\mathcal{A}_t^{\mathbb{P}^0}$.

Proof. Using the same argument as in the proof of Proposition 2, the problem given in (15) can be approximated by a problem of the form

$$\begin{aligned} \min_{x, w} \quad & w \\ \text{s.t. } \quad & x_0^\top S_0 - w \leq 0 \\ \mathbb{E}^{\mathbb{P}^0} \left[Z_t^{i,j} \cdot (-x_{t-1}^\top S_t + x_t^\top S_t + C_t) \right] &\leq 0 \quad \forall t = 1, \dots, T-1; \forall i = 1, \dots, k_t; \forall j = 1, \dots, n \\ \mathbb{E}^{\mathbb{P}^0} \left[Z_T^{i,j} \cdot (-x_{T-1}^\top S_T + C_T) \right] &\leq 0 \quad \forall i = 1, \dots, k_T; \forall j = 1, \dots, n. \end{aligned}$$

Using the Lagrangian in combination with some rearrangements, we get the equivalent formulation

$$\begin{aligned} \inf_{x_t, w} \sup_{\substack{\lambda_0 \geq 0 \\ \lambda_t^{i,j} \geq 0}} \quad & w(1 - \lambda_0) + x_0^\top \left(\lambda_0 S_0 - \sum_{i=1}^{k_1} \sum_{j=1}^n \lambda_1^{i,j} \mathbb{E}^{\mathbb{P}^0} [Z_1^{i,j} S_1] \right) \\ & + \sum_{t=1}^{T-1} \sum_{j=1}^n \mathbb{E}^{\mathbb{P}^0} \left[x_t^\top \left(S_t \sum_{i=1}^{k_t} \lambda_t^{i,j} Z_t^{i,j} - S_{t+1} \sum_{i=1}^{k_{t+1}} \lambda_{t+1}^{i,j} Z_{t+1}^{i,j} \right) \right] \\ & + \sum_{t=1}^T \sum_{j=1}^n \mathbb{E}^{\mathbb{P}^0} \left[C_t \sum_{i=1}^{k_t} \lambda_t^{i,j} Z_t^{i,j} \right]. \end{aligned} \quad (18)$$

By assumption, all probability models \mathbb{P}^0 are absolutely continuous w.r.t. $\hat{\mathbb{P}}$. Hence, by the Radon-Nikodým Theorem, for each $j = 1, \dots, n$ there exists a density process $f^j = (f_t^j)_{0 \leq t \leq T}$ such that

$$\frac{d\mathbb{P}^0}{d\hat{\mathbb{P}}} \bigg|_{\mathcal{F}_t} = \mathbb{E}^{\hat{\mathbb{P}}} \left[f_T^j \middle| \mathcal{F}_t \right] = f_t^j \quad \forall t = 0, \dots, T.$$

Rewriting all the expectations in (18) in terms of $\hat{\mathbb{P}}$ and using the tower property of the expected value, we get the following representation of the problem:

$$\begin{aligned} \inf_{x_t, w} \sup_{\substack{\lambda_0 \geq 0 \\ \lambda_t^{i,j} \geq 0}} \quad & w(1 - \lambda_0) + x_0^\top \left(\lambda_0 S_0 - \mathbb{E}^{\hat{\mathbb{P}}} \left[S_1 \sum_{j=1}^n \sum_{i=1}^{k_1} \lambda_1^{i,j} Z_1^{i,j} f_1^j \right] \right) \\ & + \sum_{t=1}^{T-1} \mathbb{E}^{\hat{\mathbb{P}}} \left[x_t^\top \left(S_t \left(\sum_{j=1}^n \sum_{i=1}^{k_t} \lambda_t^{i,j} Z_t^{i,j} f_t^j \right) - \mathbb{E}^{\hat{\mathbb{P}}} \left[S_{t+1} \left(\sum_{j=1}^n \sum_{i=1}^{k_{t+1}} \lambda_{t+1}^{i,j} Z_{t+1}^{i,j} f_{t+1}^j \right) \middle| \mathcal{F}_t \right] \right) \right] \\ & + \sum_{t=1}^T \mathbb{E}^{\hat{\mathbb{P}}} \left[C_t \left(\sum_{j=1}^n \sum_{i=1}^{k_t} \lambda_t^{i,j} Z_t^{i,j} f_t^j \right) \right] \end{aligned} \quad (19)$$

Problem (19) is bilinear and we may thus interchange the inf and the sup. Defining

$$W_t := \sum_{j=1}^n \sum_{i=1}^{k_t} \lambda_t^{i,j} Z_t^{i,j} f_t^j,$$

and carrying out explicitly the minimization in w and x_t , the unconstrained minimax problem (19) can be written in the form of a constrained maximization problem:

$$\begin{aligned} \pi_a^{\mathcal{P}_{\epsilon,k}} &= \sup_{\lambda_t^i \geq 0} \sum_{t=1}^T \mathbb{E}^{\hat{\mathbb{P}}} [C_t W_t] \\ \text{s.t. } S_t W_t &= \mathbb{E}^{\hat{\mathbb{P}}} [S_{t+1} W_{t+1} | \mathcal{F}_t] \quad \forall t = 1, \dots, T \\ W_t &\in \text{span}\{Z_t^{i,j}\} \quad \forall t = 1, \dots, T. \end{aligned} \tag{20}$$

Note that this formulation is analogous to problem (9). We thus proceed along the lines of the proof of Proposition 2 and obtain the following problem formulation:

$$\begin{aligned} \pi_a^{\mathcal{P}_{\epsilon,k}} &= \sup_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}} \left[\sum_{t=1}^T \tilde{C}_t \right] \\ \text{s.t. } \mathbb{E}^{\mathbb{Q}} [\tilde{S}_{t+1} | \mathcal{F}_t] &= \tilde{S}_t, \quad \forall t = 0, \dots, T-1 \\ \frac{d\mathbb{Q}}{d\hat{\mathbb{P}}} \Big|_{\mathcal{F}_t} &\in \text{conv} \left\{ Z_t^{i,j} f_t^j \right\}, \quad \forall t = 0, \dots, T. \end{aligned} \tag{21}$$

The limiting case $k \rightarrow \infty$ works again analogous to the proof of Prop. 2. \square

3.1 Neighborhoods for stochastic processes: nested distance balls

Ambiguity sets are typically either a finite collection of models or a neighborhood of a given baseline model. For single or two-stage models, balls w.r.t. the Wasserstein distance are commonly used (see e.g. [5, 28]). Its generalization for stochastic processes, called nested distance, is now the standard for multi-stage models. The nested distance was introduced in Pflug [24] and thoroughly described in Pflug and Pichler [25] as a concept for the distance between two discrete-time stochastic processes, which correctly takes into account both the values and information structures. If the processes are discrete in space, then the nested distance appears as a distance between scenario trees. In this paper, we use the nested distance for the construction of ambiguity sets.

Definition 5. Let $\mathbf{P} = (\Omega, \mathcal{F}, \mathbb{P})$ and $\tilde{\mathbf{P}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be two filtered probability spaces. Suppose a distance ⁶ $d : \Omega \times \tilde{\Omega} \rightarrow \mathbb{R}$ is given. The nested distance of

⁶To be precise, suppose there are random variables $\xi_t : \Omega \rightarrow \mathbb{R}^m$ and $\tilde{\xi}_t : \tilde{\Omega} \rightarrow \mathbb{R}^m$ and the distance d on $\Omega \times \tilde{\Omega}$ is inherited from ξ and $\tilde{\xi}$ by $d(\omega, \tilde{\omega}) := \sum_{t=1}^T \|\xi_t(\omega) - \tilde{\xi}_t(\tilde{\omega})\|$, where $\|\cdot\|$ denotes some norm in \mathbb{R}^m .

order $r \geq 1$ between \mathbf{P} and $\tilde{\mathbf{P}}$ is defined as

$$\begin{aligned} \mathbf{d}^{(r)}(\mathbf{P}, \tilde{\mathbf{P}}) &:= \inf_{\pi} \left(\iint_{\mathcal{F}_T \otimes \tilde{\mathcal{F}}_T} d(\omega, \tilde{\omega})^r \pi(d\omega, d\tilde{\omega}) \right)^{1/r} \\ \text{s.t. } \pi \left(A \times \tilde{\Omega} \middle| \mathcal{F}_t \otimes \tilde{\mathcal{F}}_t \right) &= \mathbb{P} \left[A \middle| \mathcal{F}_t \right] \quad A \in \mathcal{F}_T; \ t = 1, \dots, T \\ \pi \left(\Omega \times B \middle| \mathcal{F}_t \otimes \tilde{\mathcal{F}}_t \right) &= \mathbb{P} \left[B \middle| \tilde{\mathcal{F}}_t \right] \quad B \in \tilde{\mathcal{F}}_T; \ t = 1, \dots, T. \end{aligned}$$

To interpret this definition, the nested distance between two multistage probability distributions is obtained by minimizing over all nested transportation plans π (i.e. plans which are compatible with the filtration structures) from one distribution into the other. For practical applications, the discrete time and finite state space case is of particular interest. Thus, in the sequel scenario trees are used as a data structure to model the whole space and filtration. For such trees, denote by $m \prec i$ that there is a path which leads from node m to node i with a strictly positive probability (i.e. m is a (not necessarily immediate) predecessor of i). Every node is assumed to have a unique predecessor. Denote by \mathcal{N}_t and $\tilde{\mathcal{N}}_t$ the sets of nodes at stage t of the trees modelling \mathbf{P} and $\tilde{\mathbf{P}}$, respectively. For a node k and a later node i let $\mathbb{P}[i|k]$ be the conditional probability to reach i from k .

Definition 6. The nested distance of order $r \geq 1$ between two scenario trees \mathbb{P} and $\tilde{\mathbb{P}}$, is defined as

$$\begin{aligned} \mathbf{d}^{(r)}(\mathbb{P}, \tilde{\mathbb{P}})^r &:= \min_{\pi} \sum_i \sum_j \pi(i, j) d_{i,j}^r \\ \text{s.t. } \sum_{j \succ i} \pi(i, j|k, l) &= \mathbb{P}[i|k] \quad \forall i \succ k; \forall (k, l) \in (\mathcal{N}_t \times \tilde{\mathcal{N}}_t); t = 1, \dots, T \\ \sum_{i \succ k} \pi(i, j|k, l) &= \tilde{\mathbb{P}}[j|l] \quad \forall j \succ l; \forall (k, l) \in (\mathcal{N}_t \times \tilde{\mathcal{N}}_t); t = 1, \dots, T \quad (22) \\ \sum_i \sum_j \pi(i, j) &= 1 \\ \pi(i, j) &\geq 0 \quad \forall i \in \mathcal{N}_T, j \in \tilde{\mathcal{N}}_T. \end{aligned}$$

The matrix $\pi = \pi(i, j)_{i,j}$ of transportation plans and the matrix $d_{i,j} = d(\xi_i, \tilde{\xi}_j)$ carrying the distances of the paths (given some distance d on image space of the random variables) are defined on $\mathcal{N}_T \times \tilde{\mathcal{N}}_T$, the sets of nodes at stage T of the trees modelling \mathbb{P} and $\tilde{\mathbb{P}}$, respectively. The conditional joint probabilities $\pi(i, j|k, l)$ in (22) are given by

$$\pi(i, j|k, l) = \frac{\pi_{i,j}}{\sum_{i' \succ k} \sum_{j' \succ l} \pi_{i',j'}}.$$

In fact, problem (22) can be reduced to a linear program in terms of one-step conditional probabilities. Moreover, the distance can be obtained iteratively by comparing all subtrees stage by stage (see the book of Pflug and Pichler [26] for details).

Let us now investigate model ambiguity with respect to nested distance balls.

Definition 7. An ε nested distance ball around some baseline model $\hat{\mathbf{P}}$ is defined as $\mathcal{B}_\varepsilon := \{\mathbf{P} : \mathbf{d}^{(r)}(\mathbf{P}, \hat{\mathbf{P}}) \leq \varepsilon\}$.

Notice that a concrete numbering of the nodes represents the tree and is needed for algorithmic realizations. But the filtered stochastic process must be seen as represented by the equivalence class of all tree numberings, since all those trees are equivalent that differ only in permutation of their subtrees at all levels. It is evident that, for processes with at least two stages, convex structures cannot be defined with respect to a given numbering of the tree. The only convex structure for processes (and therefore for the equivalence classes of tree numberings) is the compounding structure: A convex combination of two tree distributions \mathbb{P} and $\hat{\mathbb{P}}$ is the compound distribution $\mathcal{C}(\mathbb{P}, \hat{\mathbb{P}}, \lambda)$ which takes \mathbb{P} with probability λ and $\hat{\mathbb{P}}$ with probability $1 - \lambda$. This definition can be extended to an infinite compound $\mathcal{C}(\mathbb{P}_\lambda, \Lambda)$, where λ is distributed according to Λ . For simplicity, we restrict ourselves to the fixed support case, where we are given the structure of a tree \mathbf{T} but ambiguity in the transition probabilities is assumed. We emphasize this assumption by using the notation $\mathbf{P}(\mathbf{T}, \mathbb{P})$ and $\mathbf{P}(\mathbf{T}, \hat{\mathbb{P}})$, respectively. As suggested in Analui and Pflug [1], a convexification of the nonconvex ambiguity set

$$\mathcal{P}_\varepsilon := \{ \mathbf{P}(\mathbf{T}, \mathbb{P}) : \mathbb{P} \in \mathcal{B}_\varepsilon \}$$

can be done by considering the *convex hull by compounding*

$$\overline{\mathcal{P}}_\varepsilon = \{ \mathcal{C}(\mathbf{P}(\mathbf{T}, \cdot), \Lambda) : \Lambda \text{ is a probability measure on } \mathcal{B}_\varepsilon \}.$$

On this extended set, the ambiguity problem is convex.

4 Algorithmic Solution and Numerical Studies

One may summarize the results of the previous sections in the following way: If the martingale measure is not unique (incomplete market), then typically there is a positive bid-ask spread in the (pointwise) replication model. This spread does also exist in the acceptability model. However, if the acceptability functional is the $\mathbb{AV@R}_\alpha$, then by changing α we can transition between the replication model ($\alpha = 0$) and the expectation model ($\alpha = 1$). At least in the latter model, but possibly earlier (i.e. for $\alpha < 1$) there is no bid-ask spread and thus a unique price. On the other hand, model ambiguity widens the bid-ask spread: The more models are considered resp. the larger the radius of the ambiguity set, the wider is the bid-ask spread. Let us now investigate some examples to demonstrate this behaviour.

Example 1. Consider the simple ternary tree in Figure 1a. Since infinitely many equivalent martingale measures can be constructed on this tree, there is a considerable bid-ask spread for the pointwise replication model, which corresponds to the $\mathbb{AV@R}_\alpha$ model with $\alpha = 0$. However, by increasing α for both contract sides, the bid-ask spread gets monotonically smaller. For $\alpha = 1$, there is no bid-ask spread, since all martingale measures coincide in their expectation and both buyer and seller only consider expectation in their valuation. Figure 2a visualizes this behaviour for the price of a call option struck at 95: The bid-price increases with α , while the ask-price decreases. For $\alpha = 1$ they coincide. The latter effect is due to the underlying tree model itself being a martingale process. For non-martingale models, the bid-ask spread typically closes for values of α

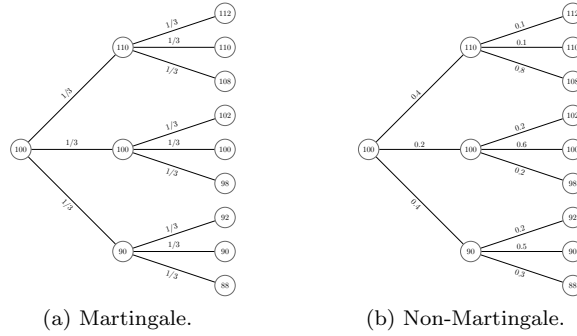


Figure 1: Sample Ternary Trees.

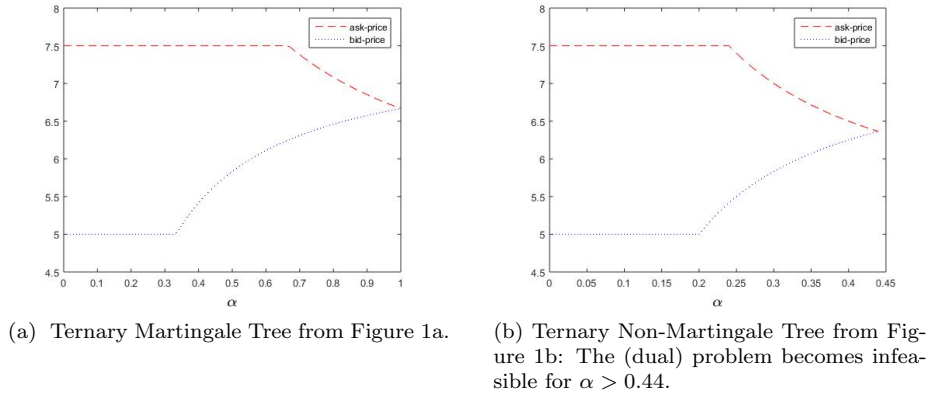
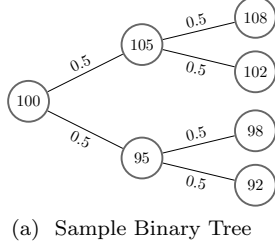


Figure 2: An incomplete market situation: The bid-ask spread as a function of the acceptability level α .

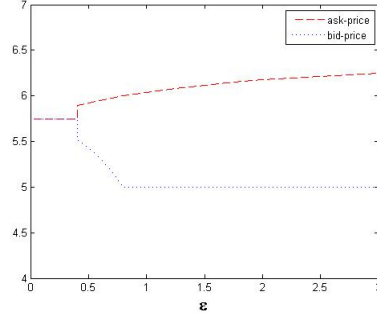
that are smaller than 1. For instance, the corresponding spread for the tree in Figure 1b is shown in Figure 2b. In this case, the spread closes for a value of $\bar{\alpha} \approx 0.44$. Note that for a value of $\alpha > \bar{\alpha}$, no feasible martingale measure exists.

Numerical solution. From a computational perspective, it is easy to see from the dual problems (4) and (13) that (for a tree model) the problem boils down to a linear program (LP). The primal problem can be implemented as an LP using a gimmick described in Pflug [23] to implement $\mathbb{AV@R}$ optimal portfolio allocation problems.

Example 2. In contrast, one may consider a binary tree model, which may - of course - carry only one martingale measure. In such a model, the change of acceptability levels does not change the price, since also under weakened acceptability the price is determined by a martingale measure, namely the unique one (in case α is small enough such that it is feasible). However, in an ambiguity situation, a bid-ask spread may appear, since there are typically many martingale measures in ambiguity sets. We illustrate this behaviour on the simple binary tree of Figure 3a. We consider nested balls around this baseline tree, where we keep the uniform distribution of the scenarios, but allow the values of the process to change. The result for a call option struck at 95 can be seen in Figure



(a) Sample Binary Tree



(b) A bid-ask spread opens when the ambiguity radius gets bigger.

Figure 3: A complete market situation: The bid-ask spread as a function of the ambiguity level ε .

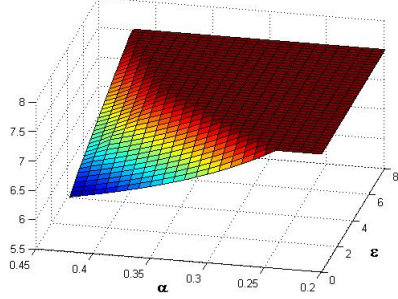
3b. While there is a unique price for small radii ε of the nested distance ball, an increasing bid-ask spread appears for larger values of ε .

Numerical solution. The problem with changing node values (i.e. atoms of the probability distribution), is nonlinear in general: in fact, we face a nonconvex problem even if the transition probabilities are kept fixed. The results in Figure 3b are based on the standard nonlinear solver of a commercial software package⁷, which finds (local) optima for our (extremely) small instance of a problem.

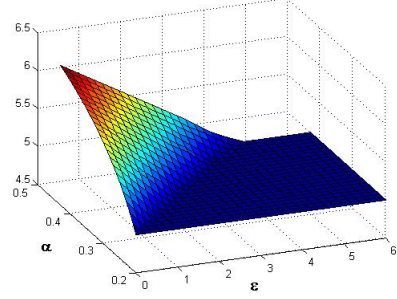
Remark. Let us comment on the structure of this problem. For single stage problems with a convex objective function and fixed probabilities, no mass-splitting is part of the optimal transportation plan (cf. Esfahani and Kuhn [5]). For the multistage case, the same is conjectured. Then, it is a combinatoric problem of how many transportation plan *designs* (plans from which node to transport to which node, regardless of how much mass is to be transported) exist. For such a fixed design, the problem becomes indeed convex. From this perspective, the feasible set can be seen as a union of convex sets. A global (brute force) algorithm would work by checking all the possible designs and evaluating each one to optimality (i.e. solving a convex optimization problem), finally picking the global optimum. However, obviously this approach is practically untractable due to the combinatorial growth of the number of transportation plan designs with the number of nodes.

Example 3. Let us now examine the simultaneous effects of acceptability and ambiguity, using again the ternary tree of Figure 1b as a baseline model and evaluating a call option struck at 95. In general, the ask-price decreases with the acceptability level α , but increases with the ambiguity radius ε . Figure 4a illustrates the particular dependency of the ask-price on α and ε for this example. On the other hand, the bid-price increases with α , but decreases with ε . The dependency of the bid-price on α resp. ε is shown in Figure 4b. Finally, Figure 4c visualizes the behaviour of the size of the bid-ask spread. Note that the plots in Figure 4 only consider values for α such that a price exists for $\varepsilon = 0$ (and thus for all values of $\varepsilon \geq 0$). In fact, for larger values of α , a price may

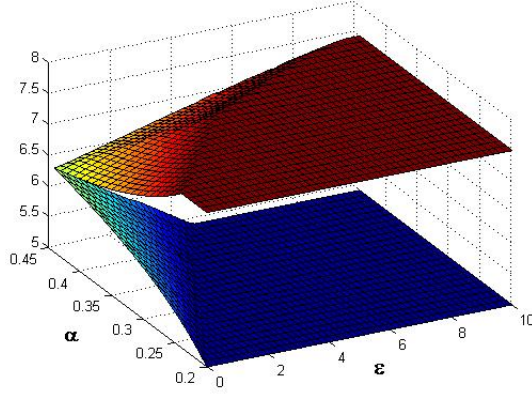
⁷MATLAB 8.5 (R2015a), The MathWorks Inc., Natick, MA, 2015.



(a) Ask-price surface.



(b) Bid-price surface.



(c) Bid-ask spread surface

Figure 4: Bid and ask prices as a function of the acceptability level α and the ambiguity radius ε

not exist (i.e. problem (15) resp. (16) is unbounded) for small ε 's, however for ε large enough it does.

Numerical solution. Due to the fact that a nested distance ball typically contains infinitely many models, there are infinitely many constraints present in (15) (and (16)) and we thus face a semi-infinite problem. To find an approximate solution in such a situation, Analui and Pflug [1] suggested the technique of successive convex programming (extending the original idea of Pflug and Wozabal [28] for Wasserstein balls in robust portfolio selection to the multi-stage case). The general idea is to split the problem into two separate problems: an *outer* problem (finding the optimal strategy considering all the probability models currently contained in \mathcal{P}_ε) and an *inner* problem (finding a new worst case model given the currently optimal strategy). One starts with an ambiguity set \mathcal{P}_ε containing only the baseline model $\hat{\mathbb{P}}$. The outer and the inner problem are then solved in alternating order, while after each iteration the new worst case model is added to \mathcal{P}_ε . The algorithm terminates when there is no improvement in the optimal objective value of the outer problem for consecutive iteration

steps. In the present case, the outer problem is an LP, as explained in Example 1. However, unfortunately the inner problem

$$\begin{aligned} \min_{\mathbb{P}} \text{AV@R}^{\mathbb{P}}(x_{T-1}^{\top} S_T - C_T) \\ \text{s.t. } \mathbf{d}^{(t)}(\mathbb{P}, \hat{\mathbb{P}}) \leq \varepsilon \end{aligned} \quad (23)$$

which needs to be solved, is highly nonlinear; in fact even nonconvex in general. We thus restrict ourselves to trees around $\hat{\mathbb{P}}$ which share exactly the same node values but rather differ in terms of the transition probabilities. Moreover, we follow the idea of [1] to exploit the structure of the nested distance and approximate the solution of (23) by an iterative procedure in terms of transportation subkernels.

Definition. Consider a scenario tree with node set \mathcal{N}_t at stage t . Let $k, l \in \mathcal{N}_t$ and $i' \in k+$, $j' \in l+$. The *transportation subkernel* $K_t(j'|i' : k, l)$ is defined as

$$K_t(j'|i' : k, l) := \frac{\pi(i', j'|k, l)}{\sum_{j'} \pi(i', j'|k, l)},$$

where $\pi(+, +|-,-)$ denotes conditional joint probabilities.

It follows directly from the definition that for a (feasible) transportation π from some tree $\mathbb{P}^{(1)}$ to some other tree $\mathbb{P}^{(2)}$, it holds that

$$\pi(i', j'|k, l) = K_t(j'|i' : k, l) \cdot \mathbb{P}^{(1)}[i'|k],$$

and thus the relation

$$\begin{aligned} \mathbb{P}^{(2)}[j_t|j_{t-1}] &= \sum_{i_t \in i_{t-1}+} \pi(i_t, j_t|i_{t-1}, j_{t-1}) \\ &= \sum_{i_t \in i_{t-1}+} K_{t-1}(j_t|i_t : i_{t-1}, j_{t-1}) \cdot \mathbb{P}^{(1)}[i_t|i_{t-1}], \end{aligned}$$

offers a direct connection between the conditional marginal distributions. This relation provides a tractable method to construct new (worst case) models out of a given baseline model. In particular, we used the following algorithm to approximate the solution of the inner problem:

Start with the current worst case model $\hat{\mathbb{P}} \circ K_0^{(\text{old})} \circ \dots \circ K_{T-1}^{(\text{old})}$.

For $t = 1$ to T solve:

$$\begin{aligned} \min_{K_{t-1}} \text{AV@R}_{\alpha}^{\hat{\mathbb{P}} \circ K_0^{(\text{new})} \circ \dots \circ K_{t-2}^{(\text{new})} \circ K_{t-1} \circ K_t^{(\text{old})} \circ \dots \circ K_{T-1}^{(\text{old})}}(x_{T-1}^{\top} S_T - C_T) \\ \text{s.t. } \mathbf{d}^{(t)}(\hat{\mathbb{P}} \circ K_0^{(\text{new})} \circ \dots \circ K_{t-2}^{(\text{new})} \circ K_{t-1} \circ K_t^{(\text{old})} \circ \dots \circ K_{T-1}^{(\text{old})}, \hat{\mathbb{P}}) \leq \varepsilon \end{aligned}$$

and update $K_{t-1}^{(\text{old})}$ to $K_{t-1}^{(\text{new})}$.

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